

# The generalized non-linear Schrödinger model on the interval

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## Abstract

The generalized  $(1+1)$ -D non-linear Schrödinger (NLS) theory with particular integrable boundary conditions is considered. More precisely, two distinct types of boundary conditions, known as soliton preserving (SP) and soliton non-preserving (SNP), are implemented into the classical  $gl_N$  NLS model. Based on this choice of boundaries the relevant conserved quantities are computed and the corresponding equations of motion are derived. A suitable quantum lattice version of the boundary generalized NLS model is also investigated. The first non-trivial local integral of motion is explicitly computed, and the spectrum and Bethe Ansatz equations are derived for the soliton non-preserving boundary conditions.

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# 1 Introduction

After various investigations (cf. below for detailed references) it is now well established that any integrable system (with finite or infinite degrees of freedom) based on the higher rank algebras  $gl_N$  or  $\mathcal{U}_q(gl_N)$  may be endowed with two distinct types of integrable<sup>4</sup> boundary conditions. These boundary conditions are known as soliton preserving (SP), traditionally recognized in the framework of integrable quantum spin chains (finite number of degrees of freedom) [1]–[8], and soliton non-preserving (SNP) originally introduced in the context of classical integrable field theories (infinite number of degrees of freedom) [9], and further investigated in [10, 11, 12]. SNP boundary conditions have been also introduced and studied for integrable quantum lattice systems [6], [13]–[16], and their quantum integrability was first shown in [13]. From the physical point of view the SP boundary conditions oblige a particle-like excitation to reflect to itself: no multiplet changing occurs. The SNP boundary conditions, on the other hand, force an excitation to reflect to its ‘conjugate’, namely to an excitation carrying the conjugate representation. From the algebraic perspective the two types of boundary conditions are associated with two distinct algebras, i.e. the reflection algebra [1, 2] and the twisted Yangian respectively [17, 18].

The study of the underlying algebraic structures defined by the Yang-Baxter and reflection equations is in general of great consequence, both at classical and quantum level, not only for integrable systems per se, but also for other relevant problems in theoretical physics. For instance, in the context of the AdS/CFT correspondence [19] it is known that from the string theory side the relevant classical integrable model is a sigma model (see e.g. [20, 21]), that is a field theory with infinite degrees of freedom. From the quantum gauge theory side on the other hand the loop contributions are apparently given by integrable quantum 1-D lattice models with a finite number of degrees of freedom [22, 23]. As a consequence, a crucial point would be the formulation of a discrete-quantum counterpart of the aforementioned classical sigma model. In this respect, the knowledge of the discrete-quantum Lax operator would facilitate the derivation of the relevant Hamiltonian, and of the other charges in involution, as well as of the exact nested Bethe Ansatz equations. In fact, up to date only the asymptotic forms of the *would-be-exact* Bethe Ansatz equations are known (see e.g. [23]). Thus a rigorous derivation of these equations would undoubtedly be of great physical significance, as proven when corrections to the asymptotic regime are available [24, 25].

In the frame of classical continuum theories the SNP boundary conditions have been primarily investigated up to now [9]–[12]. Therefore it is of great importance to further analyze the other set of boundary conditions, i.e. the SP ones within this context. In the present study we examine both SP and SNP boundary conditions for the classical generalized

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<sup>4</sup>They are defined as those boundary conditions that preserve the integrability of the system.

NLS model, and by using Hamiltonian methods [26] we derive the relevant integrals of motion, and also specify the corresponding classical equations of motion. Note that in [27] the quantum  $gl_N$  NLS model on the half line was studied, based on the reflection algebra (i.e. SP boundaries), primarily from the point of view of the underlying symmetry algebras. It should be stressed that although in most classical continuum theories the SNP boundary conditions have been analyzed (see e.g. [9, 11, 12]) this is the first time that such boundary conditions are implemented within the generalized NLS model. Here we consider the classical NLS model as a simple example, however our main motivation is to search for all possible boundary conditions in other classical theories such as affine Toda field theories, principal chiral models and others. From the viewpoint of quantum lattice models on the other hand the extensively analyzed boundary conditions are the SP ones, thus we focus here mostly on the SNP case for a lattice version of NLS. In particular, we consider a suitable lattice version of the NLS model [28, 29] with SNP boundary conditions, and we derive the exact spectrum and the corresponding Bethe Ansatz equations.

The outline of this article is as follows: in the next section we present the generic algebraic setting for classical models on the full line and on the interval. More precisely we introduce the classical Yang-Baxter equation and the underlying algebra for the system on the full line. In the case of a system on the interval we distinguish two types of boundary conditions based on the classical versions of reflection algebra (SP) and twisted Yangian (SNP). Next the NLS model on the full line is reviewed and an explicit derivation of the local integrals of motion by solving the auxiliary linear problem [26] is presented. In section 3 being guided by the same logic and adopting Sklyanin's formulation [2] we derive the integrals of motion of the  $gl_N$  NLS system on the interval with SP and SNP boundary conditions. Moreover, the corresponding classical equations of motion are obtained for various boundary conditions. In addition the usual NLS model with a reflecting impurity is investigated in the same spirit. In section 4 a suitable lattice version of the NLS model is investigated. After a brief review on the model with periodic boundary conditions we deal with the model with open boundaries. First the spectrum and Bethe Ansatz equations are derived for the usual lattice NLS. Finally, the SNP boundary conditions are considered for the generalized NLS system. The first non-trivial local integral of motion is explicitly specified for particular choice of boundary conditions, and the spectrum and Bethe Ansatz equations are deduced for the simplest boundaries.

## 2 The general setting

The line of attack which will be adopted for the study of the  $gl_N$  NLS model with integrable boundaries is based on the solution of the so called auxiliary linear problem [26]. It is therefore necessary to recall at least the basics regarding this formulation. Let  $\Psi$  be a solution of the following set of equations

$$\frac{\partial \Psi}{\partial x} = \mathbb{U}(x, t, \lambda) \Psi \quad (2.1)$$

$$\frac{\partial \Psi}{\partial t} = \mathbb{V}(x, t, \lambda) \Psi \quad (2.2)$$

with  $\mathbb{U}$ ,  $\mathbb{V}$  being in general  $n \times n$  matrices with entries functions of complex valued fields, their derivatives, and the spectral parameter  $\lambda$ . The monodromy matrix from (2.1) may be then written as:

$$T(x, y, \lambda) = \mathcal{P}exp\left\{\int_y^x \mathbb{U}(x', t, \lambda) dx'\right\}. \quad (2.3)$$

The fact that  $T$  is a solution of equation (2.1) will be extensively used subsequently for obtaining the relevant integrals of motion. Compatibility conditions of the two differential equation (2.1), (2.2) lead to the zero curvature condition

$$\dot{\mathbb{U}} - \mathbb{V}' + [\mathbb{U}, \mathbb{V}] = 0, \quad (2.4)$$

giving rise to the corresponding classical equations of motion of the system under consideration.

There exists an alternative description, known as the  $r$  matrix approach (Hamiltonian formulation). In this picture the underlying classical algebra is manifest in analogy to the quantum case as will become quite transparent later. Let us first recall this method for a general classical integrable system on the full line. The existence of the classical  $r$ -matrix, satisfying the classical Yang-Baxter equation

$$\left[r_{12}(\lambda_1 - \lambda_2), r_{13}(\lambda_1) + r_{23}(\lambda_2)\right] + \left[r_{13}(\lambda_1), r_{23}(\lambda_2)\right] = 0, \quad (2.5)$$

guarantees the integrability of the classical system. Indeed, consider the operator  $T(x, y, \lambda)$  satisfying

$$\left\{T_1(x, y, t, \lambda_1), T_2(x, y, t, \lambda_2)\right\} = \left[r_{12}(\lambda_1 - \lambda_2), T_1(x, y, t, \lambda_1)T_2(x, y, t, \lambda_2)\right]. \quad (2.6)$$

Making use of the latter equation one may readily show for a system in full line:

$$\left\{\ln \text{tr}\{T(x, y, \lambda_1)\}, \ln \text{tr}\{T(x, y, \lambda_2)\}\right\} = 0 \quad (2.7)$$

i.e. the system is integrable, and the charges in involution –local integrals of motion– may be attained by the expansion of the object  $\ln \text{tr}\{T(x, y, \lambda)\}$ , based essentially on the fact that  $T$  also satisfies (2.1).

Our main aim here is to consider the  $gl_N$  NLS model on the interval. For this purpose we follow the line of action described in [26], but using Sklyanin's formulation for the system on the interval or on the half line (see also [30] for the sine Gordon on the half line). We briefly describe this process below for any classical integrable system on the interval. In this case one has to derive a modified transition matrix  $\mathcal{T}$ , based on Sklyanin's formulation and satisfying the following Poisson bracket algebras  $\mathbb{R}$ ,  $\mathbb{T}$ , i.e. classical versions of the reflection algebra and twisted Yangian respectively. It will be convenient for our purposes here to introduce some useful notation. Let

$$\begin{aligned} r_{12}^*(\lambda) &= r_{12}(\lambda) \quad \text{for SP}, & r_{12}^*(\lambda) &= \bar{r}_{12}(\lambda) = V_1 r_{12}^{t_1}(-\lambda) V_1 \quad \text{for SNP} \\ \hat{T}(\lambda) &= T^{-1}(-\lambda) \quad \text{for SP}, & \hat{T}(\lambda) &= V T^t(-\lambda) V \quad \text{for SNP} \\ V &= \text{antid}(1, 1, \dots, 1) \quad \text{or} \quad V = \text{antid}(i, -i, \dots, -i) \quad \text{for } n \text{ even only} \end{aligned} \quad (2.8)$$

In general  $V$  can be any matrix such that  $V^2 = \mathbb{I}$ , for instance  $V = \mathbb{I}$  is also a possible choice (see e.g. [9]). Then the defining relations describing the classical reflection algebra and the twisted Yangian respectively, may be written in the following compact form<sup>5</sup>:

$$\begin{aligned} \left\{ \mathcal{T}_1(\lambda_1), \mathcal{T}_2(\lambda_2) \right\} &= \left[ r_{12}(\lambda_1 - \lambda_2), \mathcal{T}_1(\lambda_1) \mathcal{T}_2(\lambda_2) \right] \\ &+ \mathcal{T}_1(\lambda_1) r_{12}^*(\lambda_1 + \lambda_2) \mathcal{T}_2(\lambda_2) - \mathcal{T}_2(\lambda_2) r_{12}^*(\lambda_1 + \lambda_2) \mathcal{T}_1(\lambda_1). \end{aligned} \quad (2.9)$$

To construct the generating function of the integrals of motion one also needs  $c$ -number representations of the algebra  $\mathbb{R}$  ( $\mathbb{T}$ ) satisfying (2.9) for SP and SNP respectively, and also

$$\left\{ K_1^\pm(\lambda_1), K_2^\pm(\lambda_2) \right\} = 0. \quad (2.10)$$

The modified transition matrices, compatible with the corresponding algebras  $\mathbb{R}$ ,  $\mathbb{T}$  are given by the following expressions [2]:

$$\mathcal{T}(x, y, t, \lambda) = T(x, y, t, \lambda) K^-(\lambda) \hat{T}(x, y, t, \lambda) \quad (2.11)$$

and the generating function of the involutive quantities is defined as

$$t(x, y, t, \lambda) = \text{tr}\{K^+(\lambda) \mathcal{T}(x, y, t, \lambda)\} \quad (2.12)$$

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<sup>5</sup>Note that the classical versions of the reflection equation and the twisted Yangian are provided in general by more involved expressions for generic  $r$  matrices. In the present study we focus on  $r$  matrices satisfying  $r_{12}(\lambda) = r_{21}(\lambda)$  ( $\bar{r}_{12}(\lambda) = \bar{r}_{21}(\lambda)$ ), and in this case (2.9), are valid.

Due to (2.9) it can be shown that (3.10)

$$\left\{ t(x, y, t, \lambda_1), t(x, y, t, \lambda_2) \right\} = 0, \quad \lambda_1, \lambda_2 \in \mathbb{C}. \quad (2.13)$$

Technical details on the proof of classical integrability are provided in Appendix A.

By expanding  $\ln t(\lambda)$  in powers of  $\lambda^{-1}$  one recovers the local integrals of motion of the considered system, and this is achieved in the subsequent sections. Among the local integrals of motion there exist naturally the Hamiltonian, which also provides information regarding the corresponding equations of motion. This is in fact the formulation we are going to assume for the NLS system on the half line, although an alternative strategy would be to derive the modified Lax pair, compatible with the boundary conditions chosen, and hence the associated equations of motion (see e.g. [9]). Nonetheless, the rigorous derivation of the modified Lax pair is essentially based on the existence of local integrals of motion [26], therefore the viewpoint adopted here is arguably the most natural.

### 3 Classical local integrals of motion

The main objective in this section is to solve the auxiliary linear problem for the generalized NLS model on the interval. Before however we proceed to the study of the model on the interval we shall briefly review the system on the full line. In any case, these results will be relevant for the boundary case as well.

#### 3.1 The generalized NLS on the full line

We shall hereafter focus on the  $gl_N$  NLS model. Consider the classical  $r$  matrix

$$r(\lambda) = \frac{\mathbb{P}}{\lambda} \quad \text{where} \quad \mathbb{P} = \sum_{i,j=1}^N E_{ij} \otimes E_{ji} \quad (3.1)$$

$\mathbb{P}$  is the permutation operator, and  $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$ . The Lax pair for the generalized NLS model is given by the following expressions [26]:

$$\mathbb{U} = \mathbb{U}_0 + \lambda \mathbb{U}_1, \quad \mathbb{V} = \mathbb{V}_0 + \lambda \mathbb{V}_1 + \lambda^2 \mathbb{V}_2 \quad (3.2)$$

where (see also [31])

$$\begin{aligned} \mathbb{U}_1 &= \frac{1}{2i} \left( \sum_{i=1}^{N-1} E_{ii} - E_{NN} \right), & \mathbb{U}_0 &= \sqrt{\kappa} \sum_{i=1}^{N-1} (\bar{\psi}_i E_{iN} + \psi_i E_{Ni}) \\ \mathbb{V}_0 &= i\kappa \sum_{i,j=1}^{N-1} (\bar{\psi}_i \psi_j E_{ij} - |\psi_i|^2 E_{NN}) - i\sqrt{\kappa} \sum_{i=1}^{N-1} (\bar{\psi}'_i E_{iN} - \psi'_i E_{Ni}), \\ \mathbb{V}_1 &= -\mathbb{U}_0, & \mathbb{V}_2 &= -\mathbb{U}_1 \end{aligned} \quad (3.3)$$

and  $\psi_i, \bar{\psi}_j$  satisfy<sup>6</sup>:

$$\left\{ \psi_i(x), \psi_j(y) \right\} = \left\{ \bar{\psi}_i(x), \bar{\psi}_j(y) \right\} = 0, \quad \left\{ \psi_i(x), \bar{\psi}_j(y) \right\} = \delta_{ij} \delta(x - y). \quad (3.5)$$

From the zero curvature condition (2.4) the classical equations of motion for the generalized NLS model are entailed i.e.

$$i \frac{\partial \psi_i(x, t)}{\partial t} = - \frac{\partial^2 \psi_i(x, t)}{\partial^2 x} + 2\kappa \sum_j |\psi_j(x, t)|^2 \psi_i(x, t), \quad i, j \in \{1, \dots, N-1\}. \quad (3.6)$$

It is clear that for  $N = 2$  the equations of motion of the usual NLS model are recovered.

As already mentioned to obtain the local integrals of motion of NLS one has to expand  $T$  (2.3) in powers of  $\lambda^{-1}$  [26]. Let us consider the following ansatz for  $T$  as  $|\lambda| \rightarrow \infty$

$$T(x, y, \lambda) = (\mathbb{I} + W(x, \lambda)) \exp[Z(x, y, \lambda)] (\mathbb{I} + W(y, \lambda))^{-1} \quad (3.7)$$

where  $W$  is off diagonal matrix i.e.  $W = \sum_{i \neq j} W_{ij} E_{ij}$ , and  $Z$  is purely diagonal  $Z = \sum_{i=1}^N Z_{ii} E_{ii}$ . Also

$$Z_{ii}(\lambda) = \sum_{n=-1}^{\infty} \frac{Z_{ii}^{(n)}}{\lambda^n}, \quad W_{ij} = \sum_{n=1}^{\infty} \frac{W_{ij}^{(n)}}{\lambda^n}. \quad (3.8)$$

Inserting the latter expressions (3.8) in (2.1) one may identify the coefficients  $W_{ij}^{(n)}$  and  $Z_{ii}^{(n)}$  (see Appendix B for a detailed analysis). Notice that as  $i\lambda \rightarrow \infty$  the only non negligible contribution from  $Z^{(n)}$  comes from the  $Z_{NN}^{(n)}$  term, and is given by:

$$Z_{NN}^{(n)}(L, -L) = iL\delta_{n,-1} + \sqrt{\kappa} \sum_{i=1}^{N-1} \int_{-L}^L dx \psi_i(x) W_{iN}^{(n)}(x). \quad (3.9)$$

It is thus sufficient to determine the coefficients  $W_{iN}^{(n)}$  in order to extract the relevant local integrals of motion (see also [31]). Indeed solving (2.1) one may easily obtain:

$$\begin{aligned} W_{iN}^{(1)}(x) &= -i\sqrt{\kappa}\bar{\psi}_i(x), & W_{iN}^{(2)}(x) &= \sqrt{\kappa}\bar{\psi}_i'(x) \\ W_{iN}^{(3)}(x) &= i\sqrt{\kappa}\bar{\psi}_i''(x) - i\kappa^{\frac{3}{2}} \sum_k |\psi_k(x)|^2 \bar{\psi}_i(x), & \dots \end{aligned} \quad (3.10)$$

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<sup>6</sup>The Poisson structure for the generalized NLS model is defined as:

$$\left\{ A, B \right\} = i \sum_i \int_{-L}^L dx \left( \frac{\delta A}{\delta \psi_i(x)} \frac{\delta B}{\delta \bar{\psi}_i(x)} - \frac{\delta A}{\delta \bar{\psi}_i(x)} \frac{\delta B}{\delta \psi_i(x)} \right) \quad (3.4)$$

From the latter formulae (3.10) and taking into account (3.7), (3.9) the local integrals of motion of NLS may be readily extracted from  $\ln \text{tr} T(L, -L, \lambda)$ , i.e.

$$\begin{aligned} I_1 &= -i\kappa \int_{-L}^L dx \sum_{i=1}^{N-1} \psi_i(x) \bar{\psi}_i(x), \\ I_2 &= -\frac{\kappa}{2} \int_{-L}^L dx \sum_{i=1}^{N-1} \left( \bar{\psi}_i(x) \psi'_i(x) - \psi_i(x) \bar{\psi}'_i(x) \right), \\ I_3 &= -i\kappa \int_{-L}^L dx \sum_{i=1}^{N-1} \left( \kappa |\psi_i(x)|^2 \sum_k |\psi_k(x)|^2 + \psi'_i(x) \bar{\psi}'_i(x) \right), \quad \dots \end{aligned} \quad (3.11)$$

The corresponding familiar quantities for the generalized NLS are given by:

$$\mathcal{N} = -\frac{I_1}{i\kappa}, \quad \mathcal{P} = -\frac{I_2}{i\kappa}, \quad \mathcal{H} = -\frac{I_3}{i\kappa}, \quad (3.12)$$

and apparently

$$\{\mathcal{H}, \mathcal{P}\} = \{\mathcal{H}, \mathcal{N}\} = \{\mathcal{N}, \mathcal{P}\} = 0. \quad (3.13)$$

Again in the special case where  $N = 2$  the well known local integrals of motion for the usual NLS model on the full line are recovered.

## 3.2 The generalized NLS on the interval

After the review on the NLS on the full line we can come to our main concern, which is the evaluation of the integrals of motion after implementing integrable boundary conditions. We shall investigate subsequently both SP and SNP boundary conditions.

### 3.2.1 SP boundary conditions

Let us first consider the NLS model with SP boundary conditions. For this purpose  $c$ -number solutions of the classical reflection equation are needed. A general non-diagonal  $K$  matrix satisfying the classical reflection equation is given by (see also [6])

$$\begin{aligned} K(\lambda) &= D + A + i\xi \lambda^{-1} \\ D &= -E_{11} - c \sum_{i=2}^{N-1} E_{ii} + E_{NN}, \quad A = 2k(E_{1N} + E_{N1}), \quad c = 4k^2 + 1. \end{aligned} \quad (3.14)$$

Apparently in the case where  $k = 0$  a diagonal solution is recovered

$$K(\lambda) = - \sum_{i=1}^{N-1} E_{ii} + E_{NN} + i\xi \lambda^{-1}. \quad (3.15)$$



The more general diagonal  $K$  matrix is given by (see also e.g. [3, 5])

$$K(\lambda) = - \sum_{i=1}^l E_{ii} + \sum_{i=l+1}^N E_{ii} + i\xi\lambda^{-1}. \quad (3.16)$$

The solution (3.15) may be seen as a special case of (3.16) for  $l = N - 1$ .

Henceforth we set  $x = 0$ ,  $y = -L$ , and we focus on the case with diagonal boundaries provided by (3.15), (3.16), and  $K = \mathbb{I}$ . Also  $K^+(\lambda) = K(\lambda, \xi^+)$ ,  $K^-(\lambda) = K(-\lambda, \xi^-)$ . The quantity under expansion is

$$\begin{aligned} & \ln \operatorname{tr} \left\{ K^+(\lambda) T(0, -L, \lambda) K^-(\lambda) \hat{T}(0, -L, \lambda) \right\} = \\ & \ln \operatorname{tr} \left\{ (1 + \hat{W}(0))^{-1} K^+(\lambda) (1 + W(0)) e^{Z(0, -L)} (1 + W(-L))^{-1} K^-(\lambda) (1 + \hat{W}(-L)) e^{-\hat{Z}(0, -L)} \right\} \end{aligned} \quad (3.17)$$

where the objects with ‘hat’ are simply the same as before but now  $\lambda \rightarrow -\lambda$ . The technical details of the relevant computations are presented in Appendix C.

(I) Let us first consider the simple boundary conditions described by (3.15). Gathering all the information provided by equations (C.2), and by explicit computations concerning the  $i\lambda \rightarrow \infty$  expansion (see Appendix C, case (b) for fore details) we conclude that the integrals of motion for the NLS on the interval are given by:

$$\begin{aligned} I_1 &= -2i\kappa \int_{-L}^0 dx \sum_{i=1}^{N-1} \psi_i(x) \bar{\psi}_i(x), \\ I_3 &= -2i\kappa \int_{-L}^0 dx \sum_{i=1}^{N-1} \left( \kappa |\psi_i(x)|^2 \sum_{j=1}^{N-1} |\psi_j(x)|^2 + \psi'_i(x) \bar{\psi}'_i(x) \right) \\ &+ 2i\xi^+ \kappa \sum_{i=1}^{N-1} \psi_i(0) \bar{\psi}_i(0) - 2i\xi^- \kappa \sum_{i=1}^{N-1} \psi_i(-L) \bar{\psi}_i(-L), \quad \dots \end{aligned} \quad (3.18)$$

the quantity  $I_2$  as expected is trivial, as in the case of sine-Gordon model on the half line. Recall that in the whole line the quantity  $I_2$  corresponds essentially to the momentum, which is not a conserved quantity any more. To obtain the number of particles and the Hamiltonian we simply have to divide by  $-2i\kappa$

$$\mathcal{N} = -\frac{I_1}{2i\kappa}, \quad \mathcal{H} = -\frac{I_3}{2i\kappa} \quad \text{and} \quad \{\mathcal{H}, \mathcal{N}\} = 0. \quad (3.19)$$

It is clear that different choices of boundary conditions lead to distinct boundary contributions to the integrals of motion. A more detailed description of complicated diagonal and non diagonal boundaries is presented in Appendix C. Notice also that for  $N = 2$  the boundary Hamiltonian presented in [2] is recovered. Of course we could have considered Shewartz

boundary conditions at  $x = -L$  i.e.  $\psi(-L), \bar{\psi}(-L) = 0$  and trivial right boundary  $K^- = \mathbb{I}$  (that is the system is considered on the half line), then the boundary terms appearing at  $x = -L$  in the expressions of the integrals of motion would disappear.

As already mentioned the equations of motion will be derived based on the existence of a boundary Hamiltonian rather than on the existence of a modified Lax pair. In general, among the integrals of motion there exists a Hamiltonian (3.18) such that the relations below

$$\frac{\partial \psi_i(x, t)}{\partial t} = \left\{ \mathcal{H}(0, -L), \psi_i(x, t) \right\}, \quad \frac{\partial \bar{\psi}_i(x, t)}{\partial t} = \left\{ \mathcal{H}(0, -L), \bar{\psi}_i(x, t) \right\},$$

$$-L \leq x \leq 0 \quad (3.20)$$

give rise to the classical equations of motion. Indeed considering the Hamiltonian  $\mathcal{H}$  (3.18), (3.19), we end up with the following set of equations

$$i \frac{\partial \psi_i(x, t)}{\partial t} = - \frac{\partial^2 \psi_i(x, t)}{\partial x^2} + 2\kappa \sum_{j=1}^{N-1} |\psi_j(x, t)|^2 \psi_i(x, t)$$

$$\left( \frac{\partial \psi_i(x, t)}{\partial x} - \xi^+ \psi_i(x, t) \right)_{x=0} = \left( \frac{\partial \psi_i(x, t)}{\partial x} - \xi^- \psi_i(x, t) \right)_{x=-L} = 0,$$

$$i \in \{1, \dots, N-1\}. \quad (3.21)$$

In general the boundary Hamiltonian for the generalized NLS model may be expressed as

$$\mathcal{H} = \int_{-L}^0 dx \sum_{i=1}^{N-1} \left( \kappa |\psi_i(x)|^2 \sum_{j=1}^{N-1} |\psi_j(x)|^2 + \psi'_i(x) \bar{\psi}'_i(x) \right) + \mathcal{B} \quad (3.22)$$

where  $\mathcal{B}$  is the boundary potential. One may write the equations of motion for a generic boundary potential  $\mathcal{B}$ . It is clear that the bulk part remains intact as in (3.21), and what is only modified is the boundary conditions at  $x = 0, x = -L$  depending naturally on  $\mathcal{B}$ , i.e.

$$\left( \frac{\partial \psi_i(x, t)}{\partial x} + \frac{\partial \mathcal{B}}{\partial \bar{\psi}_i} \right)_{x=0} = \left( \frac{\partial \psi_i(x, t)}{\partial x} + \frac{\partial \mathcal{B}}{\partial \bar{\psi}_i} \right)_{x=-L} = 0 \quad (3.23)$$

Two more examples of diagonal boundaries are presented below:

(II) Consider the boundary conditions described by (3.16). The corresponding contributions to the integrals of motion due to the presence of non trivial boundaries are computed in Appendix C, case (b). In this case the boundary potential (see (C.6)) is given by

$$\mathcal{B} = - \sum_{i=l^++1}^{N-1} \left( \psi_i(0) \bar{\psi}'_i(0) + \psi'_i(0) \bar{\psi}_i(0) \right) - \xi^+ \sum_{i=1}^{l^+} \psi_i(0) \bar{\psi}_i(0)$$

$$+ \sum_{i=l^-+1}^{N-1} \left( \psi_i(-L) \bar{\psi}'_i(-L) + \psi'_i(-L) \bar{\psi}_i(-L) \right) + \xi^- \sum_{i=1}^{l^-} \psi_i(-L) \bar{\psi}_i(-L), \quad (3.24)$$

and consequently the boundary conditions for the equations of motion at  $x = 0$ ,  $x = -L$  now read as:

$$\begin{aligned} \left( \frac{\partial \psi_{j^+}(x)}{\partial x} - \xi^+ \psi_{j^+}(x) \right)_{x=0} &= \left( \frac{\partial \psi_{j^-}(x)}{\partial x} - \xi^- \psi_{j^-}(x) \right)_{x=-L} = 0, \quad j^\pm \in \{1, \dots, l^\pm\} \\ \psi_{j^+}(0) &= \psi_{j^-}(-L) = 0, \quad j^\pm \in \{l^\pm + 1, \dots, N-1\}. \end{aligned} \quad (3.25)$$

The previous case (I) may be seen as a special case of the more general diagonal boundary conditions by setting  $l^\pm = N-1$ . Ultimately, one would like to investigate the SP boundary conditions in the context of affine Toda field theories, something that has not been achieved up to date. In this case, it is naturally anticipated that the corresponding equations of motion should explicitly depend on the parameters  $\xi^\pm$ ,  $l^\pm$ , contrary to the case analyzed in [9], where no extra *free* parameters associated to the boundaries occur. It is also worth stressing that in the context of integrable spin chains the integers  $l^\pm$  appear explicitly in the corresponding Hamiltonian as well as in the associated symmetry of the model. More precisely, it was shown in [5] that the open spin chain with diagonal boundary conditions associated to integers  $l^\pm = l$  is  $gl_l \otimes gl_{N-l}$  invariant (or  $\mathcal{U}_q(gl_l) \otimes \mathcal{U}_q(gl_{N-l})$  invariant in the trigonometric case). The symmetry breaking for the quantum  $gl_N$  NLS model due to presence of non trivial integrable boundaries is also discussed in [27].

(III) Finally we consider the case where  $K^\pm = \mathbb{I}$  (Appendix C, case (c)). The boundary potential in this case is

$$\mathcal{B} = - \sum_{i=1}^{N-1} \left( \psi'_i(0) \bar{\psi}_i(0) + \psi_i(0) \bar{\psi}'_i(0) \right) + \sum_{i=1}^{N-1} \left( \psi'_i(-L) \bar{\psi}_i(-L) + \psi_i(-L) \bar{\psi}'_i(-L) \right) \quad (3.26)$$

and apparently we end up with simple Dirichlet boundary conditions

$$\psi_i(0) = \psi_i(-L) = 0, \quad i \in \{1, \dots, N-1\}. \quad (3.27)$$

Note that the  $N = 2$  case in particular was investigated classically on the half line in [32], whereas the NLS equation on the interval was studied in [33].

### 3.2.2 SNP boundary conditions

Recall that in this case the object under consideration, compatible with the underlying algebra, that is the classical version of the twisted Yangian, is

$$\ln \operatorname{tr} \left\{ K^+ T(0, -L, \lambda) K^- VT^t(0, -L, -\lambda) V \right\} \quad (3.28)$$

and we choose here for simplicity  $K^\pm = \mathbb{I}$ . Note however that a generic solution of the classical twisted Yangian is given by any matrix  $K = \pm K^t$  (see [14]). We shall choose in

what follows  $V = \text{antid}(1, \dots, 1)$ . By expanding (3.28) in powers of  $\lambda^{-1}$ , along the lines described in Appendix C, explicit expressions for the integrals of motion are entailed (see (C.15), (C.16)). It is worth pointing out, bearing in mind expressions (C.15), that non-local contributions to the integrals of motion arise. This is quite an intriguing fact and it definitely merits further investigation, which however will be undertaken in a forthcoming work. Nevertheless, based on the formulas (C.15), (C.16) we may explicitly express the first non-trivial conserved quantity, which is somehow a ‘modified’ number of particles, i.e.

$$\mathcal{N}_m = \sum_{i=1}^{N-1} \int_{-L}^0 dx \, \psi_i(x) \bar{\psi}_i(x) + \int_{-L}^0 dx \, \psi_1(x) \bar{\psi}_1(x). \quad (3.29)$$

Notice that the SNP boundary modify dramatically the number of particles (see (3.11), (3.12)). Indeed, the variation due to the integrable boundary conditions is not limited to the addition of certain boundary terms to the bulk quantity, as is customary, but it gives rise to an alteration of the bulk expression itself. This is a very interesting and definitely non-conventional aspect that has not been encountered before, especially in the context of continuum integrable theories. Note finally that in the special case  $N = 2$  the ‘modified’ number of particles reduces to the usual number of particles, which is a conserved quantity for the  $sl_2$  NLS model with diagonal boundary conditions (see (3.18)).

### 3.3 The NLS model with reflecting impurity

A physically relevant example will be discussed in what follows. More precisely we shall restrict our attention to the usual NLS model, and within the framework described in the previous section we shall examine the problem of reflecting impurities attached to the ends of the system. According to [2] one may consider a more general solution of the reflection equation. Consider the classical Lax operator satisfying

$$\left\{ \mathbb{L}_1(\lambda_1), \mathbb{L}_2(\lambda_2) \right\} = \left[ r_{12}(\lambda_1 - \lambda_2), \mathbb{L}_1(\lambda_1) \mathbb{L}_2(\lambda_2) \right], \quad (3.30)$$

recall  $r(\lambda)$  is given in (3.1). For example consider the  $\mathbb{L}$  operator associated to the classical Lie algebra  $sl_2$ :

$$\mathbb{L}(\lambda) = \begin{pmatrix} \lambda + S_3 & S_1 - iS_2 \\ S_1 + iS_2 & \lambda - S_3 \end{pmatrix} \quad (3.31)$$

where apparently  $S_a$  obey

$$\{S_a, S_b\} = -i \sum_{c=1} \varepsilon_{abc} S_c \quad (3.32)$$

$\varepsilon$  being the usual antisymmetric tensor. One may easily express the later matrix in terms of canonical variables  $(x, X)$ , i.e.

$$\mathbb{L}(\lambda) = \begin{pmatrix} \lambda + xX - \rho & -x^2X + 2\rho x \\ X & \lambda - xX + \rho \end{pmatrix}. \quad (3.33)$$

Degenerate cases of the matrix above are for instance the Toda chain and the DST model (see e.g. [34] and references therein) with Lax operators given by

$$\mathbb{L}^{Toda}(\lambda) = \begin{pmatrix} \lambda + X & -e^x \\ e^{-x} & 0 \end{pmatrix}, \quad \mathbb{L}^{DST}(\lambda) = \begin{pmatrix} \lambda + xX & -x \\ X & -1 \end{pmatrix} \quad (3.34)$$

Consider the following generating function of the integrals of motion

$$\ln \text{tr} \left\{ \mathbb{K}^+(\lambda) T(-L, 0, \lambda) \mathbb{K}^-(\lambda) \hat{T}(-L, 0, \lambda) \right\} \quad (3.35)$$

$\mathbb{K}$  is a generic ‘dynamical’ type solution of the classical reflection equation [2], i.e.

$$\mathbb{K}^\pm(\lambda) = \mathbb{L}(\lambda - \Theta) K^\pm(\lambda) \mathbb{L}^{-1}(-\lambda - \Theta) \quad (3.36)$$

$\mathbb{L}$  can be any solution of (3.30),  $K^\pm$  are any  $c$ -number solutions of classical reflection equation. Note that the Poisson brackets for  $\mathbb{K}$  in the classical reflection equation are considered with respect to the canonical variables  $(x, X)$ . Here we shall deal with a simple example, that is  $K^\pm = \mathbb{I}$  and  $\Theta^\pm = 0$  and  $\mathbb{L}$  given by (3.33) (for simplicity set  $\rho = 0$ ) then it is clear that

$$\mathbb{K}^\pm(\lambda) = \pm\lambda + 2 \begin{pmatrix} x^\pm X^\pm & -(x^\pm)^2 X^\pm \\ X^\pm & x^\pm X^\pm \end{pmatrix}. \quad (3.37)$$

Finally the boundary contribution to the Hamiltonian is given by (see also Appendix B)

$$I_3^{(b)} = \frac{1}{3}h_1^3 - h_1h_2 + h_3 + \frac{1}{3}\bar{h}_1^3 - \bar{h}_1\bar{h}_2 + \bar{h}_3 + 2i\kappa\psi(0)\bar{\psi}'(0) - 2i\kappa\psi(-L)\bar{\psi}'(-L). \quad (3.38)$$

$$\begin{aligned} h_0 &= 1, \quad h_1 = 2Z^+, \quad h_2 = 2\kappa\psi(0)\bar{\psi}(0) - 2i\sqrt{\kappa}\left(x^+Z^+\psi(0) + (x^+)^{-1}Z^+\bar{\psi}(0)\right), \\ h_3 &= 2i\kappa\psi(0)'\bar{\psi}(0) + 4\kappa Z^+\psi(0)\bar{\psi}(0) + 2\sqrt{\kappa}\left((x^+)^{-1}Z^+\bar{\psi}'(0) + x^+Z^+\psi'(0)\right) \end{aligned} \quad (3.39)$$

where  $Z^\pm = x^\pm X^\pm$ . Analogous expressions to (3.39) are given for  $\bar{h}$ , in particular  $\bar{h}_n = (-)^n h_n$ :  $0 \rightarrow -L$ ,  $(x^+, X^+) \rightarrow (x^-, X^-)$ . Based on the latter expressions the boundary part of the Hamiltonian may be deduced. Indeed, bearing in mind that the boundary potential is given by  $\mathcal{B} = -\frac{I_3^{(b)}}{2i\kappa}$  and taking into account (3.38), (3.39) we conclude

$$\mathcal{B} = b(x^+, Z^+, 0) - b(x^-, Z^-, -L) \quad (3.40)$$

where we define

$$\begin{aligned} \mathbf{b}(\mathbf{x}, Z, x) &= -\frac{2}{\sqrt{\kappa}} \left( \mathbf{x} Z^2 \psi(x) + \mathbf{x}^{-1} Z^2 \bar{\psi}(x) \right) - \frac{1}{i\sqrt{\kappa}} \left( \mathbf{x}^{-1} Z \bar{\psi}'(x) + \mathbf{x} Z \psi'(x) \right) \\ &\quad - \left( \psi'(x) \bar{\psi}(x) + \psi(x) \bar{\psi}'(x) \right) - \frac{4}{3i\kappa} Z^3 \end{aligned} \quad (3.41)$$

and as expected the boundary contribution of the Hamiltonian is solely expressed in terms of the canonical variables  $\mathbf{x}^\pm$ ,  $\mathbf{X}^\pm$  as well as the boundary values of the fields and their derivatives (see also [35] for a similar treatment of the classical sine-Gordon model).

## 4 A quantum lattice version of NLS

### 4.1 Review on periodic lattice NLS

Let us first present the general algebraic framework associated to the discrete quantum version of the NLS model, introduced and studied for the periodic case in [28, 29]. In the quantum level the key object as is well known is the  $\mathbb{L}$  operator satisfying:

$$R_{12}(\lambda_1 - \lambda_2) \mathbb{L}_{1n}(\lambda_1) \mathbb{L}_{2n}(\lambda_2) = \mathbb{L}_{2n}(\lambda_2) \mathbb{L}_{1n}(\lambda_1) R_{12}(\lambda_1 - \lambda_2) \quad (4.1)$$

where the  $R$ -matrix associated to the  $gl_N$  Yangian is

$$R(\lambda) = \lambda - i\kappa \mathbb{P}, \quad (4.2)$$

and obeys of course the Yang-Baxter equation [36, 37], i.e.

$$R_{12}(\lambda_1 - \lambda_2) R_{13}(\lambda_1) R_{23}(\lambda_2) = R_{23}(\lambda_2) R_{13}(\lambda_1) R_{12}(\lambda_1 - \lambda_2). \quad (4.3)$$

We shall focus in this and the subsequent section in the simplest  $sl_2$  Yangian. In this case a simple solution of equation (4.1) is given by (on a detailed description of the underlying algebra see e.g. [28, 29])

$$\mathbb{L}_{0n}(\lambda) = \begin{pmatrix} 1 - i\Delta\lambda + \Delta^2\kappa\phi_n\psi_n & -i\Delta\sqrt{\kappa}\phi_n \\ i\Delta\sqrt{\kappa}\psi_n & 1 \end{pmatrix} \quad (4.4)$$

where the  $\psi_n, \phi_n$  satisfy canonical commutation relations

$$[\psi_n, \phi_m] = \frac{1}{\Delta} \delta_{nm}. \quad (4.5)$$

In fact this solution may be thought of as the quantum version of the NLS. Indeed the classical limit of the  $L$  operator (4.4) gives  $\mathbb{U}$  (3.3) (for further details see [28]). Set  $\psi_n = \int_{x_n}^{x_n+\Delta} dx \psi(x)$  then as  $\Delta \rightarrow 0$

$$\mathbb{L}(\lambda) = 1 + \Delta \tilde{\mathbb{U}}(x, \lambda) + \mathcal{O}(\Delta^2), \quad \text{where} \quad \tilde{\mathbb{U}}(x, \lambda) = \begin{pmatrix} i\lambda & \sqrt{\kappa}\phi(x) \\ \sqrt{\kappa}\psi(x) & 0 \end{pmatrix} \quad (4.6)$$

note that  $\phi(x) = \bar{\psi}(x)$ , and  $\tilde{\mathbb{U}}$  is equivalent to  $\mathbb{U}$  (3.3) of NLS up to a gauge transformation i.e.

$$\mathbb{U} = h\tilde{\mathbb{U}}h^{-1} + h_x h^{-1}, \quad h = e^{-ix\frac{\lambda}{2}}. \quad (4.7)$$

It is more convenient for our purposes here to use  $\mathbb{L}$  with a rescaled spectral parameter matrix. Let us multiply (4.4) by  $\frac{i\lambda}{\Delta}$  and also set  $\zeta = \frac{i}{\lambda}$  then the rescaled  $\mathbb{L}$  matrix may be written as:

$$\mathbb{L}_{0n}(\lambda) = \begin{pmatrix} 1 + \frac{\mathbb{N}_n}{\Delta}\zeta & -i\zeta\sqrt{\kappa}\phi_n \\ i\zeta\sqrt{\kappa}\psi_n & \frac{\zeta}{\Delta} \end{pmatrix} \quad (4.8)$$

$\mathbb{N}_n = 1 + \kappa\Delta^2\phi_n\psi_n$ . For the special value  $\zeta = 0$  the  $\mathbb{L}$  operator reduces to a projector

$$\mathbb{L}(\zeta = 0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \quad (4.9)$$

Due to the fact that the algebra (4.1) is equipped with a coproduct one may build tensorial representations and construct a spin chain like system with periodic boundary conditions, by introducing the quantities

$$T_0(\lambda) = \mathbb{L}_{0L} \dots \mathbb{L}_{01}, \quad \text{and} \quad t(\lambda) = \text{tr}_0 T_0(\lambda) \quad (4.10)$$

with  $T(\lambda)$  being essentially the quantum analogous of (2.3) and apparently by virtue of (4.1)

$$[t(\lambda), t(\lambda')] = 0. \quad (4.11)$$

By expanding  $\ln t(\zeta)$  around  $\zeta = 0$  we find the corresponding involutive quantities exactly as in the classical case [28]. It is easy to see that due to (4.9)  $t(0) = 1$ , then one finds (for more details we refer the reader to [28])

$$\ln t(\zeta) = \zeta\kappa C_1 + \zeta^2\kappa C_2 + \zeta^3\kappa C_3 + \dots, \quad [C_n, C_m] = 0 \quad (4.12)$$

with

$$\begin{aligned} C_1 &= \frac{1}{\Delta\kappa} \sum_{n=1}^L \mathbb{N}_n, \\ C_2 &= \sum_{n=1}^L p_n = \sum_{n=1}^L \left( \phi_{n+1}\psi_n - \frac{1}{2\kappa\Delta^2} \mathbb{N}_n^2 \right), \\ C_3 &= \frac{1}{\Delta} \sum_{n=1}^L h_n = \sum_{n=1}^L \left( \phi_{n+1}\psi_{n-1} - (\mathbb{N}_n + \mathbb{N}_{n+1})\phi_{n+1}\psi_n + (3\kappa\Delta^2)^{-1}\mathbb{N}_n^3 \right). \end{aligned} \quad (4.13)$$

From the latter objects one may derive lattice versions of the classical quantities (3.12),

$$\begin{aligned}\mathcal{N} &= (C_1 - L)|_{\Delta \rightarrow 0} \rightarrow \int dx \phi(x)\psi(x), \\ \mathcal{P} &= (C_2 + \frac{L}{2\kappa\Delta^2})|_{\Delta \rightarrow 0} \rightarrow \frac{1}{2} \int dx (\phi_x\psi - \phi\psi_x) \\ \mathcal{H} &= -C_3 + \frac{L}{3\kappa\Delta^2}|_{\Delta \rightarrow 0} \rightarrow \int dx (\phi_x\psi_x + \kappa(\phi\psi)^2).\end{aligned}\tag{4.14}$$

Notice that the expressions above are symmetric to  $\psi$ ,  $\phi$  so one can set  $\phi = \bar{\psi}$  and obtain the familiar expressions for the NLS system (3.12). Note also that the existence of an obvious pseudo-vacuum allows the implementation of Bethe ansatz techniques [38, 28], however our aim here is to extend such computations in the case of the integrable open spin chain, which is discussed in the subsequent sections.

## 4.2 Open lattice NLS

We come now to the case with open boundary conditions, which is our main concern. The underlying algebra in this case is defined by the reflection equation [1]

$$R_{12}(\lambda_1 - \lambda_2)K_1(\lambda_1)R_{21}(\lambda_1 + \lambda_2)K_2(\lambda_2) = K_2(\lambda_2)R_{21}(\lambda_1 + \lambda_2)K_1(\lambda_1)R_{12}(\lambda_1 - \lambda_2).\tag{4.15}$$

The tensorial type solutions of the reflection equation as well known is given by [2]

$$\mathcal{T}_0(\lambda) = T_0(\lambda) K_0^-(\lambda, \xi^-, c^-) T_0^{-1}(-\lambda)\tag{4.16}$$

$K^-(\lambda, \xi^-, k^-)$  is  $c$ -number solution of the reflection equation with the most general form given by [39]

$$K^\pm(\lambda) = \lambda\sigma^z \pm i\xi^\pm + 2k^\pm\lambda(\sigma^+ + \sigma^-).\tag{4.17}$$

Note that the explicit expression of  $\mathbb{L}^{-1}(-\lambda) = \hat{\mathbb{L}}(\lambda)$  is given by:

$$\hat{\mathbb{L}}_{0n}(\lambda) = \begin{pmatrix} 1 & i\Delta\sqrt{\kappa}\phi_n \\ -i\Delta\sqrt{\kappa}\psi_n & i\Delta\lambda + \mathbb{N}_n - \kappa\Delta \end{pmatrix}.\tag{4.18}$$

The corresponding generating function of the conserved quantities of the open system is

$$t(\lambda) = tr_0 \left\{ K_0^+(\lambda, \xi^+, k^+) \mathcal{T}_0(\lambda) \right\}\tag{4.19}$$

$K^+(\lambda, \xi^+, k^+) = K(-\lambda - i, \xi^+, k^+)^t$ ,  $K$  is again a  $c$  number solution of the reflection equation. And due to (4.15) it is clear that integrability of the quantum system is ensured i.e.

$$\left[ t(\lambda), t(\lambda') \right] = 0, \quad \lambda, \lambda' \in \mathbb{C}.\tag{4.20}$$



In the remaining of this section we specify the spectrum of the lattice NLS model with diagonal boundary conditions by means of the Bethe ansatz technique [38]. Focusing on diagonal boundaries should be sufficient given that in [6] the spectral equivalence between systems with diagonal and non diagonal boundaries was shown by means of appropriate gauge transformations, but only for spin chains associated to the fundamental representation of  $sl_2$ . Presumably there exist suitable gauge transformations for the system under consideration, such that the spectral equivalence is guaranteed. We shall further comment on this point on a separate publication. When both boundaries are diagonal there exists an obvious reference state for the transfer state

$$\Omega = \bigotimes_{n=1}^N \varpi_n \quad \text{with} \quad \psi_n \varpi_n = 0. \quad (4.21)$$

Based on this observation one may in a straightforward manner derive the spectrum and the corresponding Bethe ansatz equations. We provide directly the results avoiding the technical details (for a detailed description we refer the reader to [2]). The spectrum of the transfer matrix is given by

$$\Lambda(\lambda) = g(\lambda)b_1(\lambda) \prod_{j=1}^M \frac{(\lambda - \lambda_j + i\kappa)(\lambda + \lambda_j)}{(\lambda - \lambda_j)(\lambda + \lambda_j - i\kappa)} + h(\lambda)b_2(\lambda) \prod_{j=1}^M \frac{(\lambda - \lambda_j - i\kappa)(\lambda + \lambda_j - 2i\kappa)}{(\lambda - \lambda_j)(\lambda + \lambda_j - i\kappa)} \quad (4.22)$$

where we define

$$\begin{aligned} g(\lambda) &= (-i\lambda\Delta + 1)^L, \quad h(\lambda) = (i\lambda\Delta + 1 + \kappa\Delta)^L, \\ b_1(\lambda) &= \frac{\lambda - i\kappa}{\lambda - \frac{i\kappa}{2}}(\lambda + i\xi^-)(-\lambda + i\xi^+), \quad b_2(\lambda) = \frac{\lambda}{\lambda - \frac{i\kappa}{2}}(-\lambda + i\xi^- + i\kappa)(\lambda - i\kappa + i\xi^+) \end{aligned} \quad (4.23)$$

The corresponding Bethe ansatz equations arising as analyticity conditions on the spectrum are:

$$\frac{g(\lambda_i + \frac{i\kappa}{2})}{h(\lambda_i + \frac{i\kappa}{2})} \frac{b_1(\lambda_i + \frac{i\kappa}{2})}{b_2(\lambda_i + \frac{i\kappa}{2})} = - \prod_{j=1}^M \frac{\lambda_i - \lambda_j - i\kappa}{\lambda_i - \lambda_j + i\kappa} \frac{\lambda_i + \lambda_j - i\kappa}{\lambda_i + \lambda_j + i\kappa}. \quad (4.24)$$

Notice that although we deal with an open spin chain and one would expect a leading order of  $2L$ , we see a leading order of  $L$  exactly as in the periodic case. The same phenomenon occurs in the boundary lattice Liouville model [40], and is presumably associated to the degenerate nature of the  $\mathbb{L}$  matrix (see also similar comments in [34]). It should be emphasized that the Bethe ansatz equations (4.24) are of particular significance given that their thermodynamic analysis yields for instance consequential information regarding the corresponding bulk as well as boundary exact  $S$  matrices of the model.

## 5 The generalized lattice NLS

We shall deal in what follows with the lattice quantum version of the  $gl_N$  NLS model. Recall that the  $gl_N$  Yangian  $R$  matrix, solution of the Yang–Baxter equation (4.3), is given in (4.2). The relevant  $\mathbb{L}$  operator in this case is given by (see also [28])

$$\mathbb{L}(\lambda) = \left(-\frac{i\lambda}{\kappa} + \sum_{j=1}^{N-1} \phi^{(j)}\psi^{(j)}\right)E_{11} + \sum_{j=2}^N E_{jj} + \sum_{j=2}^N (\phi^{(j-1)}E_{1j} + \psi^{(j-1)}E_{j1}). \quad (5.1)$$

Notice that here we set implicitly  $i\Delta\sqrt{\kappa} = 1$ . It will be also useful for the following to define

$$\hat{\mathbb{L}}(\lambda) = V_1 \mathbb{L}^{t_1}(-\lambda + i\kappa\rho) V_1, \quad \rho = \frac{N}{2} \quad (5.2)$$

we choose here  $V = \text{antid}(1, \dots, 1)$ , which gives rise to the following explicit form:

$$\hat{\mathbb{L}}(\lambda) = \left(\frac{i\lambda}{\kappa} + \rho + \sum_{j=1}^{N-1} \phi^{(j)}\psi^{(j)}\right)E_{NN} + \sum_{j=1}^{N-1} E_{jj} + \sum_{j=2}^N (\phi^{(j-1)}E_{\bar{j}N} + \psi^{(j-1)}E_{N\bar{j}}) \quad (5.3)$$

where  $\bar{j} = N - j + 1$ . Recall that in general we could have chosen any  $V$  such that  $V^2 = \mathbb{I}$ . We shall focus hereafter in the case of SNP boundary conditions, given that they are not so widely known compared to the SP ones, especially in the context of integrable lattice models. The main objective in this section is to derive the exact spectrum and the corresponding Bethe ansatz equations. Note that in the SNP case the underlying algebra is defined by the following relation, (twisted Yangian, see e.g. [17])

$$R_{12}(\lambda_1 - \lambda_2) K_1(\lambda_1) \bar{R}_{21}(\lambda_1 + \lambda_2) K_2(\lambda_2) = K_2(\lambda_2) \bar{R}_{12}(\lambda_1 + \lambda_2) K_1(\lambda_1) R_{21}(\lambda_1 - \lambda_2) \quad (5.4)$$

and in analogy to the classical case we define

$$\bar{R}(\lambda) = V_1 R_{12}^{t_1}(-\lambda + i\rho\kappa) V_1. \quad (5.5)$$

The generating function of the quantum integrals of motion in this case is defined as:

$$t(\lambda) = \text{tr} \left\{ K^+(\lambda) T(\lambda) K^-(\lambda) \hat{T}(\lambda) \right\}, \quad \text{with} \\ T(\lambda) = \mathbb{L}_{0L}(\lambda) \dots \mathbb{L}_{01}(\lambda), \quad \hat{T}(\lambda) = \hat{\mathbb{L}}_{01}(\lambda) \dots \hat{\mathbb{L}}_{0L}(\lambda) \quad (5.6)$$

$K^\pm$  are  $c$ -number solutions of the twisted Yangian (5.4). In fact, it was shown in [14] that any matrix  $K = \pm K^t$  is a solution of the twisted Yangian. In Appendix D an explicit computation of local integrals of motion for particular boundary conditions is presented. Based on these findings we present the explicit form of the boundary momentum in the case where  $K^\pm = V$ , i.e.

$$\mathcal{P}_d = -\frac{i\kappa}{2} \left( \sum_{n=1}^L \mathbb{N}_n^2 - 2 \sum_{n=1}^{L-1} \sum_{j=1}^{N-1} \psi_n^{(j)} \phi_{n+1}^{(j)} + \sum_{j=1}^{N-1} (\psi_L^{(j)} \psi_L^{(j)} + \phi_1^{(j)} \psi_1^{(j)}) \right). \quad (5.7)$$

We could have of course considered  $K^\pm \propto \mathbb{I}$ , which is the case will be examined subsequently, but for the sake of simplicity we considered the aforementioned boundary conditions, which from a technical point of view are much easier to deal with. By comparing the bulk momentum given in Appendix D (D.4) with (5.7) we conclude that the periodic terms in (D.4) are replaced essentially by the last two boundary terms in (D.7), whereas the bulk part remains intact. Following the logic of (4.14), it is expected that the expression (5.7) should be a regularization of the continuum generalized NLS model momentum with particular SNP boundary conditions, exactly as it happens for the periodic NLS model (indeed compare the bulk continuum expression (3.11) with the discrete periodic analogous (D.6)). Comments on higher conserved charges may be also found in Appendix D.

To deduce the spectrum and Bethe ansatz equations of the generalized NLS model with SNP boundary conditions we shall restrict our attention to another simple case i.e.  $K^\pm = \mathbb{I}$ . Again the spectral equivalence between systems with diagonal and non-diagonal boundaries is discussed in [14, 16], for spin chains associated to the fundamental representation of  $gl_N$ . The first step toward the diagonalization of the transfer matrix (5.6) is the derivation of a reference state. Indeed, in this case there exists an obvious reference state, that is

$$\Omega = \bigotimes_{n=1}^L \varpi_n : \quad \psi_n^{(i)} \varpi_n = 0, \quad n \in \{1, \dots, L\}, \quad i \in \{1, \dots, N-1\}. \quad (5.8)$$

The corresponding eigenvalue may be easily derived using the fact that  $\mathbb{L}$ ,  $\hat{\mathbb{L}}$  and consequently  $T$ ,  $\hat{T}$  satisfy

$$\hat{T}_1(\lambda_1) \bar{R}_{12}(\lambda_1 + \lambda_2) T_2(\lambda_2) = T_2(\lambda_2) \bar{R}_{12}(\lambda_1 + \lambda_2) \hat{T}_1(\lambda_1). \quad (5.9)$$

Taking into account the latter relation we conclude that the actions of the transfer matrix on the pseudo vacuum provides the following eigenvalue:

$$\Lambda^{(0)}(\lambda) = a^L(\lambda) g_1(\lambda) + \sum_{n=2}^{N-1} g_n(\lambda) + \bar{b}^L(\lambda) g_N(\lambda) \quad (5.10)$$

where we define

$$\begin{aligned} a(\lambda) &= -\frac{i\lambda}{\kappa}, & \bar{b}(\lambda) &= \frac{i\lambda}{\kappa} + \rho \\ g_n(\lambda) &= \frac{\lambda - \frac{i\kappa}{2}(\rho - 1)}{\lambda - \frac{i\kappa}{2}}, & 1 \leq n < \frac{N+1}{2} \\ g_{\frac{N+1}{2}} &= 1, & N \text{ odd} \\ g_l &= g_{\bar{l}}(-\lambda + i\kappa\rho), & \frac{N+1}{2} < l \leq N. \end{aligned} \quad (5.11)$$

The functions  $g_n$  are essentially ‘boundary’ contributions to the spectrum.

To determine the general eigenvalue form we shall adopt the analytical Bethe ansatz formulation [41]. The basic assumption within this framework is that the structure of any eigenvalue is similar to the pseudo-vacuum eigenvalue i.e.

$$\Lambda(\lambda) = a^L(\lambda) g_1(\lambda) A_1(\lambda) + \sum_{n=2}^{N-1} g_n(\lambda) A_n(\lambda) + \bar{b}^L(\lambda) g_N(\lambda) A_N(\lambda). \quad (5.12)$$

The so called dressing functions  $A_n$  may be explicitly determined by imposing certain physical and algebraic requirements, such as analyticity, crossing, etc. We do not give all the details of the formulation, (for a more detailed description of the process we refer the reader to [4, 6, 14]), but the explicit expressions for the dressing functions are given by:

$$\begin{aligned} A_1(\lambda) &= \prod_{j=1}^{M^{(1)}} \frac{\lambda + \lambda_j^{(1)} + \frac{i\kappa}{2}}{\lambda + \lambda_j^{(1)} - \frac{i\kappa}{2}} \frac{\lambda - \lambda_j^{(1)} + \frac{i\kappa}{2}}{\lambda - \lambda_j^{(1)} - \frac{i\kappa}{2}} \\ A_{l+1} &= \prod_{j=1}^{M^{(l)}} \frac{\lambda + \lambda_j^{(l)} - \frac{il\kappa}{2} - i\kappa}{\lambda + \lambda_j^{(l)} - \frac{il\kappa}{2}} \frac{\lambda - \lambda_j^{(l)} - \frac{il\kappa}{2} - i\kappa}{\lambda - \lambda_j^{(l)} - \frac{il\kappa}{2}} \\ &\quad \prod_{j=1}^{M^{(l+1)}} \frac{\lambda + \lambda_j^{(l+1)} - \frac{il\kappa}{2} + \frac{i\kappa}{2}}{\lambda + \lambda_j^{(l+1)} - \frac{il\kappa}{2} - \frac{i\kappa}{2}} \frac{\lambda - \lambda_j^{(l+1)} - \frac{il\kappa}{2} + \frac{i\kappa}{2}}{\lambda - \lambda_j^{(l+1)} - \frac{il\kappa}{2} - \frac{i\kappa}{2}}, \quad 1 \leq l < \frac{N-1}{2} \\ A_l(\lambda) &= A_{\bar{l}}(-\lambda + i\kappa\rho), \quad \frac{N-1}{2} < l \leq N \end{aligned} \quad (5.13)$$

and in particular for  $N = 2n + 1$

$$\begin{aligned} A_{n+1}(\lambda) &= \prod_{j=1}^{M^{(n)}} \frac{\lambda + \lambda_j^{(n)} - \frac{in\kappa}{2} - i\kappa}{\lambda + \lambda_j^{(n)} - \frac{in\kappa}{2}} \frac{\lambda - \lambda_j^{(n)} - \frac{in\kappa}{2} - i\kappa}{\lambda - \lambda_j^{(n)} - \frac{in\kappa}{2}} \\ &\quad \times \prod_{j=1}^{M^{(n+1)}} \frac{\lambda + \lambda_j^{(n+1)} - \frac{in\kappa}{2} + \frac{i\kappa}{2}}{\lambda + \lambda_j^{(n+1)} - \frac{in\kappa}{2} - \frac{i\kappa}{2}} \frac{\lambda - \lambda_j^{(n+1)} - \frac{in\kappa}{2} + \frac{i\kappa}{2}}{\lambda - \lambda_j^{(n+1)} - \frac{in\kappa}{2} - \frac{i\kappa}{2}}. \end{aligned} \quad (5.14)$$

Finally Bethe ansatz equations follow as analyticity requirements upon the spectrum, and they are written explicitly as:

(i)  $N=2n+1$ :

$$\begin{aligned} \left(a(\lambda_i^{(1)} + \frac{i\kappa}{2})\right)^L \delta_{l1} + (1 - \delta_{l1}) &= - \prod_{j=1}^{M^{(l)}} e_2(\lambda_i^{(l)} - \lambda_j^{(l)}) e_2(\lambda_i^{(l)} + \lambda_j^{(l)}) \\ &\quad \times \prod_{j=1}^{M^{(l+\tau)}} e_{-1}(\lambda_i^{(l)} - \lambda_j^{(l+\tau)}) e_{-1}(\lambda_i^{(l)} + \lambda_j^{(l+\tau)}) \\ l &= 1, \dots, n-1 \end{aligned}$$

$$\begin{aligned}
e_{-\frac{1}{2}}(\lambda_i^{(n)}) &= - \prod_{j=1}^{M^{(n)}} e_2(\lambda_i^{(n)} - \lambda_j^{(n)}) e_2(\lambda_i^{(n)} + \lambda_j^{(n)}) e_{-1}(\lambda_i^{(n)} - \lambda_j^{(n)}) e_{-1}(\lambda_i^{(n)} + \lambda_j^{(n)}) \\
&\times \prod_{j=1}^{M^{(n-1)}} e_{-1}(\lambda_i^{(n)} - \lambda_j^{(n-1)}) e_{-1}(\lambda_i^{(n)} + \lambda_j^{(n-1)})
\end{aligned} \tag{5.15}$$

with  $\tau = \pm 1$ ,  $M^{(N+1)} = 0$  and define  $e_n(\lambda) = \frac{\lambda - \frac{i n \kappa}{2}}{\lambda + \frac{i n \kappa}{2}}$ .

(ii) **N=2n**: In this case the Bethe ansatz equations for  $l = 1, \dots, n-1$  are the same as in the previous case. What is only modified is the last set of equations, which takes the form:

$$e_{-\frac{1}{2}}(\lambda_i^{(n)}) = - \prod_{j=1}^{M^{(n)}} e_2(\lambda_i^{(n)} - \lambda_j^{(n)}) e_2(\lambda_i^{(n)} + \lambda_j^{(n)}) \prod_{j=1}^{M^{(n-1)}} e_{-1}^2(\lambda_i^{(n)} - \lambda_j^{(n-1)}) e_{-1}^2(\lambda_i^{(n)} + \lambda_j^{(n-1)}). \tag{5.16}$$

Such type of boundary conditions were first introduced in [13] for the  $sl_3$  spin chain, whereas generalizations investigated in [6, 14] from the physical (Bethe ansatz) as well as the algebraic point of view. The associated symmetries were studied in detail in [14, 15, 16], while in [15] both SP and SNP boundary conditions were examined in parallel. The interesting observation is that the RHS of the equations above in the case where  $N = 2n+1$  coincide with the ones associated to the  $osp(1|2n)$  algebra. In any case the Bethe ansatz equations (5.15) are somehow ‘folded’ compared to the usual  $gl_N$  ones. This is expected given that folding occurs at the algebraic level as well (Dynkin diagrams), and only the subalgebra invariant under charge conjugation survives after the implementation of these rather unconventional boundary conditions (see also relevant comments in [14]). The case of SP boundaries can be also treated along the same lines, and the entailed spectrum and Bethe ansatz equations will have the usual  $gl_N$  structure (the expressions are omitted here for brevity). More precisely the LHS of the Bethe ansatz equations will have exactly the same form as the usual  $gl_N$  BAE for an open chain (see e.g. [3, 5, 6]), while the RHS will depend explicitly on the actions of the diagonal entries of the  $\mathbb{L}$ ,  $\mathbb{L}^{-1}$  on the pseudo-vacuum.

## 6 Discussion

To summarize, SP and SNP boundary conditions were studied for the classical generalized NLS model, and the boundary integrals of motion as well as the relevant classical equations of motion were explicitly derived. This was the first time, to our knowledge, that SNP boundaries were implemented in the context of the generalized NLS model. Nevertheless,

there are still several open questions especially regarding the locality of some of the integrals of motion for particular choices of left/right boundaries, which however will be left for future investigations. In the same spirit the usual ( $sl_2$ ) NLS model with reflecting impurities was also analyzed. Moreover, a suitable lattice version of the generalized NLS model was considered and the SNP boundary conditions were implemented, given that in general they are much less studied in this context. For this choice of boundary conditions we were able to specify the first non-trivial local integral of motion i.e. the ‘boundary momentum’. We also derived the spectrum and Bethe Ansatz equations for the simplest left/right boundaries.

Although SP boundary conditions are somehow the obvious ones in the framework of lattice integrable models, they have not been really considered up to now in classical continuum integrable theories. Therefore, it will be our next goal to impose the SP boundary conditions to other well known classical systems such as (massless) affine Toda field theories, principal chiral models, etc. In addition, the investigation of the generalized (m)KdV hierarchies [42] with integrable boundaries is another very interesting direction to pursue together with their quantization into an appropriate lattice version (see e.g. [43] and references therein). Once this point is clarified, the study of the underlying dynamical symmetries constrained by integrable boundary conditions could be discussed in full generality along the lines described in [44], and this would definitively shed new light on the character of the different integrable boundary conditions. More precisely, it would be of great consequence to examine how the so called hidden symmetries constructed in [44] are modified in the presence of non-trivial integrable boundaries, and in particular in the case of (quantum) twisted Yangians. Finally, an interesting direction to pursue is the explicit derivation of the modified Lax pairs by means of the ‘boundary’ integrals of motion. We hope to report on all these issues in forthcoming publications.

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## A Appendix

In this appendix the classical integrability for models with both types of boundary conditions, SP and SNP, is reviewed. The first step in order to prove the classical integrability is to show that the quantities introduced in (2.11) are indeed representations of the algebras defined

by (2.9). To achieve this we shall need in addition to (2.6) the following set of algebraic relations emerging essentially from (2.6), i.e.

$$\begin{aligned}\left\{\hat{T}_1(\lambda_1), \hat{T}_2(\lambda_2)\right\} &= r_{12}(\lambda_1 - \lambda_2) \hat{T}_1(\lambda_1) \hat{T}_2(\lambda_2) - T_1(\lambda_1) \hat{T}_2(\lambda_2) r_{12}(\lambda_1 - \lambda_2) \\ \left\{T_1(\lambda_1), \hat{T}_2(\lambda_2)\right\} &= T_1(\lambda_1) r_{12}^*(\lambda_1 + \lambda_2) \hat{T}_2(\lambda_2) - \hat{T}_2(\lambda_2) r_{12}^*(\lambda_1 + \lambda_2) T_1(\lambda_1) \\ \left\{\hat{T}_1(\lambda_1), T_2(\lambda_2)\right\} &= \hat{T}_1(\lambda_1) r_{12}^*(\lambda_1 + \lambda_2) T_2(\lambda_2) - T_2(\lambda_2) r_{12}^*(\lambda_1 + \lambda_2) \hat{T}_1(\lambda_1) \quad (\text{A.1})\end{aligned}$$

Our aim now is to show that (2.9) are satisfied by (2.8):

$$\begin{aligned}\left\{\mathcal{T}_1(\lambda_1), \mathcal{T}_2(\lambda_2)\right\} &= \left\{T_1(\lambda_1)K_1^-(\lambda_1)\hat{T}_1(\lambda_1), T_2(\lambda_2)K_2^-(\lambda_2)\hat{T}_2(\lambda_2)\right\} = \dots \\ &= T_1(\lambda_1)T_2(\lambda_2)\left(-K_1(\lambda_1)K_2(\lambda_2)r_{12}(\lambda_1 - \lambda_2) + r_{12}(\lambda_1 - \lambda_2)K_1(\lambda_1)K_2(\lambda_2)\right. \\ &\quad \left.+ K_1(\lambda_1)r_{12}^*(\lambda_1 + \lambda_2)K_2(\lambda_2) - K_2(\lambda_2)r_{12}^*(\lambda_2)K_1(\lambda_1)\right)\hat{T}_1(\lambda_1)\hat{T}_2(\lambda_2) \\ &\quad + \mathcal{T}_1(\lambda_1)r_{12}^*(\lambda_1 + \lambda_2)\mathcal{T}_2(\lambda_2) - \mathcal{T}_2(\lambda_2)r_{12}^*(\lambda_1 + \lambda_2)\mathcal{T}_1(\lambda_1) \\ &\quad r_{12}(\lambda_1 - \lambda_2)\mathcal{T}_1(\lambda_1)\mathcal{T}_2(\lambda_2) - \mathcal{T}_1(\lambda_1)\mathcal{T}_2(\lambda_2)r_{12}(\lambda_1 - \lambda_2) \quad (\text{A.2})\end{aligned}$$

and making use of (2.10), (A.3) and (A.1) we end up to (2.9). Recall also that  $c$ -number solutions of the above equations satisfy the following

$$\left[r_{12}(\lambda_1 - \lambda_2), K_1(\lambda_1)K_2(\lambda_2)\right] = K_2(\lambda_2)r_{12}^*(\lambda_1 + \lambda_2)K_1(\lambda_1) - K_1(\lambda_1)r_{12}^*(\lambda_1 + \lambda_2)K_2(\lambda_2), \quad (\text{A.3})$$

which is equivalent to (2.10).

We may now show exploiting (2.9) and (A.3) the classical integrability (2.13). Indeed consider the following object

$$\left\{K_1^+(\lambda_1)\mathcal{T}_1(\lambda_1), K_2^+(\lambda_2)\mathcal{T}_2(\lambda_2)\right\} \quad (\text{A.4})$$

then taking the trace in both spaces 1 and 2, and considering the defining relations (2.9), (2.10) we end up with

$$\begin{aligned}\left\{t(\lambda_1), t(\lambda_2)\right\} &= tr_{12}\left(K_1^+(\lambda_1)K_2^+(\lambda_2)r_{12}(\lambda_1 - \lambda_2)\mathcal{T}_1(\lambda_1)\mathcal{T}_2(\lambda_2)\right. \\ &\quad - K_1^+(\lambda_1)K_2^+(\lambda_2)\mathcal{T}_1(\lambda_1)\mathcal{T}_2(\lambda_2)r_{12}(\lambda_1 - \lambda_2) \\ &\quad + K_1^+(\lambda_1)K_2^+(\lambda_2)\mathcal{T}_1(\lambda_1)r_{12}^*(\lambda_1 + \lambda_2)\mathcal{T}_2(\lambda_2) \\ &\quad \left.- K_1^+(\lambda_1)K_2^+(\lambda_2)\mathcal{T}_2(\lambda_2)r_{12}^*(\lambda_1 + \lambda_2)\mathcal{T}_1(\lambda_1)\right). \quad (\text{A.5})\end{aligned}$$

Finally moving appropriately the factors of the products within the trace and using (A.3) it is straightforward to show

$$\left\{t(\lambda_1), t(\lambda_2)\right\} = 0, \quad \lambda_1, \lambda_2 \in \mathbb{C} \quad (\text{A.6})$$

and this concludes our proof. Similar arguments hold also in the case of dynamical boundaries discussed in section 3.3.

## B Appendix

We present here some technical details on the derivation of the conserved quantities for the generalized NLS model on the full line. The first step is to insert the ansatz (3.7) in equation (2.1). Then we separate the diagonal and off diagonal part and obtain the following expressions:

$$\begin{aligned} Z' &= \lambda \mathbb{U}_1 + (\mathbb{U}_0 W)^{(D)} \\ W' + W Z' &= \mathbb{U}_0 + (\mathbb{U}_0 W)^{(O)} + \lambda \mathbb{U}_1 W \end{aligned} \quad (\text{B.1})$$

where the superscripts  $(D)$ ,  $(O)$  denote the diagonal and off diagonal part of the product  $\mathbb{U}_0 W$ . Recall that  $W = \sum_{i \neq j} W_{ij} E_{ij}$ ,  $Z = \sum_i Z_{ii} E_{ii}$  then it is straightforward to obtain:

$$\begin{aligned} (\mathbb{U}_0 W)^{(D)} &= \sqrt{\kappa} \sum_{i=1}^{N-1} \left( \bar{\psi}_i W_{Ni} E_{ii} + \psi_i W_{iN} E_{NN} \right) \\ (\mathbb{U}_0 W)^{(O)} &= \sqrt{\kappa} \sum_{i \neq j, i \neq N, j \neq N} \left( \bar{\psi}_i W_{Nj} E_{ij} + \psi_i W_{ij} E_{Nj} \right). \end{aligned} \quad (\text{B.2})$$

Substituting the latter expressions (B.2) in (B.1), we obtain

$$Z(L, -L, \lambda) = -i\lambda L \left( \sum_{i=1}^{N-1} E_{ii} - E_{NN} \right) + \sqrt{\kappa} \sum_{i=1}^{N-1} \int_{-L}^L dx \left( \bar{\psi}_i W_{Ni} E_{ii} + \psi_i W_{iN} E_{NN} \right) \quad (\text{B.3})$$

And recalling that the leading contribution in the expansion of  $(\ln \text{tr} T)$ ,  $-T$  is given in (3.7)– as  $i\lambda \rightarrow \infty$  is coming from the  $Z_{NN}$  term we conclude:

$$Z_{NN}(L, -L, \lambda) = i\lambda L + \sqrt{\kappa} \sum_{i=1}^{N-1} \int_{-L}^L dx \psi_i(x) W_{iN}(x) \quad (\text{B.4})$$

Due to (B.4) it is obvious that in this case it is sufficient to derive the coefficients  $W_{iN}$  only. In any case one can show that the coefficients  $W_{ij}$  satisfy the following equations:

$$\begin{aligned} &\sum_{i \neq j} W'_{ij} E_{ij} - i\lambda \sum_{i \neq N} \left( W_{Ni} E_{Ni} - W_{iN} E_{iN} \right) + \sqrt{\kappa} \sum_{i \neq N} \left( \bar{\psi}_i W_{Ni}^2 E_{Ni} + \psi_i W_{iN}^2 E_{iN} \right) = \\ &\sqrt{\kappa} \sum_{i \neq N} \left( \bar{\psi}_i E_{iN} + \psi_i E_{Ni} \right) + \sqrt{\kappa} \sum_{i \neq j, i \neq N, j \neq N} \left( \bar{\psi}_i W_{Nj} E_{ij} + \psi_i W_{ij} E_{Nj} \right) \\ &- \sqrt{\kappa} \sum_{i \neq j, i \neq N, j \neq N} \left( \bar{\psi}_j W_{Nj} W_{ij} E_{ij} + \psi_i W_{iN} W_{jN} E_{jN} \right) \end{aligned} \quad (\text{B.5})$$

Finally setting  $W_{ij} = \sum_{n=1}^{\infty} \frac{W_{ij}^{(n)}}{\lambda^n}$  and using (B.5) we find expressions for  $W_{iN}^{(n)}$  reported in (3.10). In the case with integrable boundary conditions, we shall need in addition to (3.10)



the following objects:

$$\begin{aligned}
W_{Ni}^{(1)} &= i\sqrt{\kappa}\psi_i, & W_{Ni}^{(2)} &= -iW_{Ni}'^{(1)} + \sum_{i \neq j, i \neq N, j \neq N} W_{Nj}^{(1)} W_{ji}^{(1)}, & W_{ji}'^{(1)} &= iW_{jN}^{(1)} W_{Ni}^{(1)} \\
W_{Ni}^{(3)} &= -iW_{Ni}'^{(2)} + W_{iN}^{(1)} W_{Ni}^{(1)} W_{Ni}^{(1)} + \sum_{i \neq j, i \neq N, j \neq N} W_{Nj}^{(1)} W_{ji}^{(2)} \\
W_{ij}'^{(2)} &= iW_{iN}^{(1)} W_{Nj}^{(2)} - iW_{jN}^{(1)} W_{Nj}^{(1)} W_{ij}^{(1)}.
\end{aligned} \tag{B.6}$$

## C Appendix

In what follows we evaluate the boundary terms contributing to the Hamiltonian for right and left boundary described by the more general, diagonal and non-diagonal, solutions of the reflection equation (SP boundary conditions). Moreover, for the SNP boundary conditions we identify the corresponding integrals of motions, and we explicitly evaluate the first non-trivial charge.

### C.1 SP boundary conditions

We shall expand the generic object (3.17) keeping of course only diagonal contributions. More precisely, as in the bulk case due to the fact that the leading order is  $e^{i\lambda L}$  as  $i\lambda \rightarrow \infty$  the only non negligible part is coming from the  $E_{NN}$  terms, hence we shall only consider such contributions:

$$\begin{aligned}
\left[ (1 + \hat{W}(0, \lambda))^{-1} K^+(\lambda) (1 + W(0, \lambda)) \right]_{NN} &= \sum_{n=0}^{\infty} \frac{h_n}{\lambda^n}, \\
\left[ (1 + W(-L, \lambda))^{-1} K^-(\lambda) (1 + \hat{W}(-L, \lambda)) \right]_{NN} &= \sum_{n=0}^{\infty} \frac{\bar{h}_n}{\lambda^n} \\
\left[ Z(0, -L, \lambda) - \hat{Z}(0, -L, \lambda) \right]_{NN} &= i\lambda L + \sum_{n=1}^{\infty} (1 - (-)^n) \frac{Z_{NN}^{(n)}(0, -L)}{\lambda^n}
\end{aligned} \tag{C.1}$$

Again considering only the contribution of the term  $e^{i\lambda L}$  as  $i\lambda \rightarrow \infty$  we end up with the following expression

$$\begin{aligned}
\ln \operatorname{tr} \left\{ K^+(\lambda) T(0, -L, \lambda) K^-(\lambda) \hat{T}(0, -L, \lambda) \right\} &= \\
i\lambda L + \sum_{n=1}^{\infty} (1 - (-)^n) \frac{Z_{NN}^{(n)}(0, -L)}{\lambda^n} &+ \ln \left( \sum_{n=0}^{\infty} \frac{h_n + \bar{h}_n}{\lambda^n} + \sum_{n,m=0}^{\infty} \frac{h_n \bar{h}_m}{\lambda^{n+m}} \right)
\end{aligned} \tag{C.2}$$

Recall from (C.2) that the boundary contribution lies basically in the logarithmic function, hence one has to expand the log i.e.

$$\ln\left(\sum_{n=0}^{\infty} \frac{h_n + \bar{h}_n}{\lambda^n} + \sum_{n,m=0}^{\infty} \frac{h_n \bar{h}_m}{\lambda^{n+m}}\right) = \sum_{n=0}^{\infty} \frac{f_n}{\lambda^n} \quad (\text{C.3})$$

where  $f_n$ , provide essentially the boundary contribution to the integrals of motion (plus possible total derivatives from the bulk part) for the left right boundary respectively. The interesting observation is that the boundary contribution decouples nicely to terms associated to left and right boundary separately, i.e. no mixing occurs

$$\begin{aligned} f_1 &= h_1 + \bar{h}_1, & f_2 &= -\frac{1}{2}h_1^2 + h_2 - \frac{1}{2}\bar{h}_1^2 + \bar{h}_2, \\ f_3 &= \frac{1}{3}h_1^3 - h_1h_2 + h_3 + \frac{1}{3}\bar{h}_1^3 - \bar{h}_1\bar{h}_2 + \bar{h}_3, & \dots \end{aligned} \quad (\text{C.4})$$

(a) We first consider generic non diagonal boundary conditions described by (3.14). One can explicitly evaluate  $h_n$  for the generic case:

$$\begin{aligned} h_0 &= 1, & h_1 &= -2ik^+ \sqrt{\kappa} \left( \bar{\psi}_1(0) - \psi_1(0) \right) + i\xi^+, \\ h_2 &= 2k^+ \sqrt{\kappa} \left( \bar{\psi}'_1(0) - \psi'_1(0) \right) - 4k^{+2} \kappa \sum_{i=2}^{N-1} \psi_i(0) \bar{\psi}_i(0), \\ h_3 &= 2ik^+ \sqrt{\kappa} \left( \bar{\psi}''_1(0) - \psi''_1(0) \right) + 2\kappa i \xi^+ \sum_{j=1}^{N-1} \psi_j(0) \bar{\psi}_j(0) - 2i\kappa \sum_{j=1}^{N-1} \psi_j(0) \bar{\psi}'_j(0) \\ &\quad + 2ik^+ \kappa^{\frac{3}{2}} \sum_j |\psi_j(0)|^2 \left( \psi_1(0) - 2\bar{\psi}_1(0) \right) + 2ik^+ \kappa^{\frac{3}{2}} \bar{\psi}_1 \psi_1^2 - 4i\kappa k^{+2} \sum_{j=2}^{N-1} \left( \psi_j(0) \bar{\psi}'_j(0) + \psi'_j(0) \bar{\psi}_j(0) \right) \\ &\quad \dots \end{aligned} \quad (\text{C.5})$$

In fact it is clear from the expansion of the left and right boundary contribution that  $\bar{h}_n = (-1)^n h_n : 0 \rightarrow -L, \xi^+ \rightarrow \xi^-, k^+ \rightarrow k^-$ .

Given that there is an overall derivative from the bulk part of  $Z_3$ , giving rise to a boundary term ( $2i\kappa \sum_i \psi_i \bar{\psi}'_i$ ), we conclude that the boundary contribution to the conserved quantity  $I_3$  (i.e. the Hamiltonian) is given by

$$I_3 = f_3 + 2i\kappa \sum_{i=1}^{N-1} \bar{\psi}'_i(0) \psi_i(0) - 2i\kappa \sum_{i=1}^{N-1} \bar{\psi}'_i(-L) \psi_i(-L) = -2i\kappa \mathcal{B}, \quad (\text{C.6})$$

where  $\mathcal{B}$  is the boundary potential, and recall that  $f_3$  is given by (C.4). For diagonal boundary, terms proportional to  $k^\pm$  apparently disappear, and the Hamiltonian reduces to (3.18).

For a purely antidiagonal boundary only terms proportional to  $k^\pm$  survive, all the other terms disappear.

(b) The more general diagonal boundary conditions are described by (3.16). We associate the integers  $l^\pm$  and the free parameters  $\xi^\pm$  to the right/left boundaries. In this case from the expansions (C.1) and taking into account (3.10), (B.6) we find:

$$\begin{aligned} h_0 &= 1, \quad h_1 = i\xi^+, \quad h_2 = 2\kappa \sum_{i=l^++1}^{N-1} \psi_i(0)\bar{\psi}_i(0) \\ h_3 &= -2i\kappa \sum_{i=1}^{l^+} \psi_i(0)\bar{\psi}'_i(0) + 2i\kappa \sum_{i=l^++1}^{N-1} \psi'_i(0)\bar{\psi}_i(0) + 2i\kappa\xi \sum_{i=1}^{N-1} \psi_i(0)\bar{\psi}_i(0), \quad \dots \end{aligned} \quad (\text{C.7})$$

similar expressions of course hold for  $\bar{h}_n$ , i.e.  $\bar{h}_n = (-1)^n h_n$ :  $0 \rightarrow -L$ ,  $l^+ \rightarrow l^-$ ,  $\xi^+ \rightarrow \xi^-$ . Taking into account (C.4), (C.6) and derivative contribution from the bulk ( $2i\kappa \sum_{i=1}^{N-1} \psi_i\bar{\psi}'_i$ ) we conclude that the boundary contribution to the Hamiltonian is given by:

$$\begin{aligned} I_3 &= 2i\kappa \left[ \sum_{i=l^++1}^{N-1} \left( \psi_i(0)\bar{\psi}'_i(0) + \psi'_i(0)\bar{\psi}_i(0) \right) - \sum_{i=l^--1}^{N-1} \left( \psi_i(-L)\bar{\psi}'_i(-L) + \psi'_i(-L)\bar{\psi}_i(-L) \right) \right] \\ &+ 2i\kappa \left[ \xi^+ \sum_{i=1}^{l^+} \psi_i(0)\bar{\psi}_i(0) - \xi^- \sum_{i=1}^{l^-} \psi_i(-L)\bar{\psi}_i(-L) \right]. \end{aligned} \quad (\text{C.8})$$

Note that when  $l^\pm = N-1$  one recovers the diagonal limit of the more general case (a).

(c) Finally the case where  $K^\pm = \mathbb{I}$  may be treated in the same spirit. In this case we find that

$$h_0 = 1, \quad h_1 = 0, \quad h_2 = 2\kappa \sum_{i=1}^{N-1} \psi_i(0)\bar{\psi}_i(0), \quad h_3 = 2i\kappa \sum_{i=1}^{N-1} \psi'_i(0)\bar{\psi}_i(0), \quad \dots \quad (\text{C.9})$$

and the corresponding boundary contribution to the Hamiltonian is given by

$$I_3 = 2i\kappa \left[ \sum_{i=1}^{N-1} \left( \psi'_i(0)\bar{\psi}_i(0) + \psi_i(0)\bar{\psi}'_i(0) \right) - \sum_{i=1}^{N-1} \left( \psi'_i(-L)\bar{\psi}_i(-L) + \psi_i(-L)\bar{\psi}'_i(-L) \right) \right]. \quad (\text{C.10})$$

## C.2 SNP boundary conditions

Recall that in this case the object under consideration is given in (3.28), also we consider for simplicity  $K^\pm = \mathbb{I}$  and we choose  $V = \text{antid}(1, \dots, 1)$ . Before we proceed with the evaluation of the integrals of motion let us first introduce some useful notation

$$\hat{W}_{ij}(\lambda) = W_{\bar{j}\bar{i}}(-\lambda), \quad \hat{Z}_{ii}(\lambda) = Z_{\bar{i}\bar{i}}(-\lambda),$$

$$\text{and } (1 + W(\lambda))^{-1} = 1 + F(\lambda) \quad \text{where} \quad F(\lambda) = \sum_{n=1}^{\infty} \frac{f^{(n)}}{\lambda^n}, \quad (\text{C.11})$$

recall  $\bar{j} = N - j + 1$ , and also one may easily compute that

$$f^{(1)} = -W^{(1)}, \quad f^{(2)} = (W^{(1)})^2 - W^{(2)}, \quad \dots \quad (\text{C.12})$$

We shall need the following contributions in order to evaluate the corresponding integrals of motion:

$$\begin{aligned} (1 + \hat{W}(0, \lambda))(1 + W(0, \lambda)) &= 1 + H(\lambda), \\ (1 + W(-L, \lambda))^{-1}(1 + \hat{W}(-L, \lambda))^{-1} &= 1 + \bar{H}(\lambda). \end{aligned} \quad (\text{C.13})$$

Also bearing in mind that the leading contributions in the considered expansion, as  $i\lambda \rightarrow \infty$ , are coming from  $Z_{NN}$  and  $\hat{Z}_{ii}$  for  $i \in \{2, \dots, N\}$  we may write:

$$Z_{NN}(0, -L, \lambda) + \hat{Z}_{ii}(0, -L, \lambda) = i\lambda L + \sum_{n=1}^{\infty} \frac{Z_{NN}^{(n)}(0, -L) + (-1)^n Z_{ii}^{(n)}(0, -L)}{\lambda^n} \quad (\text{C.14})$$

Gathering all the information given above we end up with the following expression

$$\begin{aligned} \ln \text{tr} \left\{ T(0, -L, \lambda) V T^t(0, -L, -\lambda) V \right\} &= i\lambda L + \sum_{n=1}^{\infty} \frac{Z_{NN}^{(n)}(0, -L) + (-1)^n Z_{11}^{(n)}(0, -L)}{\lambda^n} \\ &+ \ln \left( 1 + H_{NN}(\lambda) + \bar{H}_{NN}(\lambda) + \sum_{i=2}^N e^{\hat{Z}_{ii}(0, -L, \lambda) - \hat{Z}_{NN}(0, -L, \lambda)} H_{iN}(\lambda) \bar{H}_{Ni}(\lambda) \right) \end{aligned} \quad (\text{C.15})$$

Finally taking into account the information provided above (C.11)–(C.15) we may express the first non-trivial integrals of motion as:

$$\begin{aligned} I_1 &= Z_{NN}^{(1)}(0, -L) - Z_{11}^{(1)}(0, -L) \\ &= -i\kappa \sum_{i=1}^{N-1} \int_{-L}^0 dx \, \psi_i(x) \bar{\psi}_i(x) - i\kappa \int_{-L}^0 dx \, \psi_1(x) \bar{\psi}_1(x) \end{aligned} \quad (\text{C.16})$$

Notice that in general due to the presence of  $Z_{11}^{(n)}$  in (C.15) non-local terms seem to arise in the higher integrals of motion, which is quite an unusual issue and shall be addressed elsewhere. Nevertheless, a straightforward computation of the higher charges, based on the explicit expression (C.15), may prove the locality or not of the higher integrals of motions. Moreover, the presence of  $Z_{ii}^{(n)}$  alters the structure of the bulk part of the integrals as well. The latter integral of motion (C.16) gives rise to a ‘modified’ number of particles,  $\mathcal{N}_m = -\frac{I_1}{i\kappa}$ .

## D Appendix

In this appendix we shall evaluate the first integrals of motion of the quantum discrete  $gl_N$  NLS model with SNP boundary conditions. This model may be also regarded as a higher rank algebraic extension of the  $sl_2$  DST model (see e.g. [34]), holding a special place between the  $gl_N$  quantum spin chains –extensions of the Heisenberg model– and the  $gl_N$  generalization of the Toda chain. To explicitly specify the local integrals of motions of the model with open boundary conditions we shall, as usual, consider the asymptotic expansion of the generating function  $\mathcal{T}(\lambda)$ . We shall focus here on the simple case where both left and right boundaries are given by  $K^\pm(\lambda) = \text{antid}(1, \dots, 1)$ , and effectively we shall expand

$$t(\lambda) = \text{tr } T(\lambda) \hat{T}(\lambda) \quad \text{where} \quad \hat{T}(\lambda) = T^t(-\lambda) \quad (\text{D.1})$$

and recall  $T(\lambda)$  is given by (5.6). Indeed after expanding in powers of  $\lambda^{-1}$  we obtain

$$\begin{aligned} T(\lambda) &\propto E_{11} + \frac{1}{\lambda} T^{(1)} + \frac{1}{\lambda^2} T^{(2)} + \mathcal{O}(\lambda^{-3}) \\ \hat{T}(\lambda) &\propto E_{11} + \frac{1}{\lambda} \hat{T}^{(1)} + \frac{1}{\lambda^2} \hat{T}^{(2)} + \mathcal{O}(\lambda^{-3}), \end{aligned} \quad (\text{D.2})$$

where the quantities  $T^{(1,2)}$ ,  $\hat{T}^{(1,2)}$  are defined below

$$\begin{aligned} T^{(1)} &= i\kappa \left( \sum_{n=1}^L \mathbb{N}_n E_{11} + \sum_{j=2}^N \phi_1^{(j-1)} E_{1j} + \sum_{j=2}^N \psi_L^{(j-1)} E_{j1} \right) \\ \hat{T}^{(1)} &= -i\kappa \left( \sum_{n=1}^L \hat{\mathbb{N}}_n E_{11} + \sum_{j=2}^N \phi_1^{(j-1)} E_{j1} + \sum_{j=2}^N \psi_L^{(j-1)} E_{1j} \right) \\ T^{(2)} &= -\kappa^2 \left( \sum_{n>m} \mathbb{N}_n \mathbb{N}_m E_{11} + \sum_{n=1}^{L-1} \sum_{j=2}^N \psi_n^{(j-1)} \phi_{n+1}^{(j-1)} E_{11} + \sum_{j=2}^N \psi_L^{(j-1)} \phi_1^{(j-1)} E_{jj} \right. \\ &\quad \left. + \sum_{n=1}^{L-1} \mathbb{N}_n \sum_{j=2}^N \psi_L^{(j-1)} E_{j1} + \sum_{j=2}^N \phi_1^{(j-1)} \sum_{n=2}^L \mathbb{N}_n E_{1j} + \sum_{j=2}^N \psi_{L-1}^{(j-1)} E_{j1} + \sum_{j=2}^N \phi_2^{(j-1)} E_{1j} \right) \\ \hat{T}^{(2)} &= -\kappa^2 \left( \sum_{n<m} \hat{\mathbb{N}}_n \hat{\mathbb{N}}_m E_{11} + \sum_{n=1}^{L-1} \sum_{j=2}^N \psi_n^{(j-1)} \phi_{n+1}^{(j-1)} E_{11} + \sum_{j=2}^N \psi_L^{(j-1)} \phi_1^{(j-1)} E_{jj} \right. \\ &\quad \left. + \sum_{n=1}^{L-1} \hat{\mathbb{N}}_n \sum_{j=2}^N \psi_L^{(j-1)} E_{1j} + \sum_{j=2}^N \phi_1^{(j-1)} \sum_{n=2}^L \hat{\mathbb{N}}_n E_{j1} + \sum_{j=2}^N \psi_{L-1}^{(j-1)} E_{1j} + \sum_{j=2}^N \phi_2^{(j-1)} E_{j1} \right) \\ &\quad \text{where} \quad \mathbb{N}_n = \sum_{j=1}^{N-1} \phi_n^{(j)} \psi_n^{(j)}, \quad \hat{\mathbb{N}}_n = \mathbb{N}_n + \rho. \end{aligned} \quad (\text{D.3})$$

In the expressions above all the lower indices denote the site of the spin chain, while the upper indices denote the component of the  $(N-1)$  dimensional vector fields.

We may easily obtain first the bulk integrals of motion by considering the expansion  $\text{tr } T(\lambda) = \sum_n \frac{I_n}{\lambda^n}$ . Indeed for the bulk case after simply taking the trace of  $T^{(1,2)}$  we obtain

$$I_1 = i\kappa \sum_{n=1}^L \mathbb{N}_n, \quad I_2 = -\kappa^2 \left( \sum_{n < m} \mathbb{N}_n \mathbb{N}_m + \sum_{n=1}^{L-1} \sum_{j=1}^{N-1} \psi_n^{(j)} \phi_{n+1}^{(j)} + \sum_{j=1}^{N-1} \psi_L^{(j)} \phi_1^{(j)} \right). \quad (\text{D.4})$$

The quantities identified as the number of particles and the momentum in the NLS model are given by the following expressions

$$\mathcal{N}_d = \frac{1}{i\kappa} I_1, \quad \mathcal{P}_d = \frac{1}{i\kappa} \left( \frac{1}{2} I_1^2 - I_2 \right) \quad (\text{D.5})$$

and more precisely

$$\mathcal{N}_d = \sum_{n=1}^L \mathbb{N}_n, \quad \mathcal{P}_d = -\frac{i\kappa}{2} \left( \sum_{n=1}^L \mathbb{N}_n^2 - 2 \sum_{n=1}^{L-1} \sum_{j=1}^{N-1} \psi_n^{(j)} \phi_{n+1}^{(j)} - 2 \sum_{j=1}^{N-1} \phi_1^{(j)} \psi_L^{(j)} \right). \quad (\text{D.6})$$

We come now to the open NLS model, and we consider the expansion of (D.1). The first charge of the open model is zero, that is the number of particles is not a conserved quantity anymore. The second charge is given by

$$I_2 = \kappa^2 \left( \sum_{n=1}^L \mathbb{N}_n^2 - 2 \sum_{n=1}^{L-1} \sum_{j=1}^{N-1} \psi_n^{(j)} \phi_{n+1}^{(j)} + \sum_{j=1}^{N-1} (\psi_L^{(j)} \psi_L^{(j)} + \phi_1^{(j)} \phi_1^{(j)}) \right) \quad (\text{D.7})$$

and corresponds to the momentum  $\mathcal{P}_d = \frac{I_2}{2i\kappa}$ , which is obviously modified due to the presence of the open boundaries. The third charge again is trivial, involving only boundary terms. We do not compute any higher conserved charges, but we may rather safely conjecture that the only non-trivial conserved charges are the even ones.

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